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KINKING OF A CRACK EMANATING FROM THE FREE SURFACE OF A BEAM UNDER PURE BENDING

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Abstract—A perturbation technique developed by Karihaloo *et al.* is employed to obtain the stress intensity factors at the tip of a kinking crack that emanates from the free surface of a beam under pure bending. Under the condition that the kink extends in the direction of vanishing K_{II} the crack path is obtained as well as a path stability condition. From conditions on K_I a material parameter r^* akin to that of Ramulu and Kobayashi's r_c is obtained. By analysis of the slope of the kinking crack a stability condition is obtained corroborating the stability condition from consideration of vanishing K_{II}. It is shown that for a beam in pure bending the nonsingular remote stress term T must be greater than some positive critical value for kinking to occur confirming the results of Sayir and Schindler.

INTRODUCTION

In the last decade several papers were published dealing with dynamic crack propagation in beam bending dealing with several aspects of the problem[1–7]. The phenomenon of crack turning and/or bifurcation in such a problem is well known having been observed by Bodner[8] and reported as early as 1972. In their study, Kinra and Kolsky[1] observed definite stages of crack propagation, the first being a fast increase of the crack velocity to some constant value then a marked decrease in the velocity at about 70–80% of the beam height. The next stage was a turn out of plane by the crack characterized by a crack velocity that was much smaller than the one in the first stage. Freund and Herrmann[2] and Adeli *et al.*[3] proposed a beam bending model to explain the first stage of crack propagation. Levy and Herrmann[4, 5] improved on that model by improving the fracture model used in [3] and by including the effects of rotary inertia and shear, thus making the model fit the experimental results in [1] in a better qualitative manner. Schindler and Sayir[6, 7] investigated the stability of the crack path in the beam, obtaining a criterion on the induced loading P for determining when the path would become unstable (i.e. the crack would deviate from its original straight path) based on a fracture model similar to that used by Levy and Herrmann[4]. Streit and Finnie[9] in their experimental analysis of double-cantilever-beam, compact-tension and center-crack specimen derived a material parameter that characterized the directional stability of the crack and related it to the constant term in the Williams[10] crack–tip stress series. Ramulu *et al.*[11, 12] in their work on dynamic crack propagation in pipes expanded on this idea and derived a parameter which also included the velocity of the crack.

In order to explain the second stage of fracture in a beam in bending, the authors have employed a perturbation technique discussed recently in papers by Cottrell and Rice[13] and Karihaloo *et al.*[14] which is based on the works of Banichuk[15], and Goldstein and Salganik[16]. The approach is quasistatic, since the crack velocity effect is small in the second stage. Though the method was employed for cracks that were semi-infinite in an infinite medium[13, 15, 16] or finite cracks in an infinite medium[14], it is hereby extended to a crack emanating from a free surface via the work of Hartranft and Sih[17]. The equations obtained are similar to those in [14] and are used to define the path of the kinking crack. It is shown that as the crack moves away from the free surface the non-singular term,

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T, in the Williams expansions[10] changes sign, from negative to positive, leading to path instability. One by-product of the method is a parameter, r^*, similar to the parameters Δa_c (Schindler and Sayir[6, 7]), r_c (Streit and Finnie[9], Ramulu *et al.*[11, 12]) and α_0 (Eftis *et al.*[18]) found by previous investigators. The term in [14] that leads to this term appears to be unaccounted for in their final calculations.

In order to solve the problem at hand we will give a short recap of the perturbation method as given in [14] and show how it must be modified to account for the free surface. Next the resulting integral equation will be solved for the stress intensity factors, $K_{\rm I}$ and $K_{\rm II}$. Applying the condition of crack propagation in the vanishing $K_{\rm II}$ direction will lead to the determination of the crack kinking path and the stability condition. The parameter r^* will then be obtained by defining an equivalent straight kink and this parameter will be compared to Ramulu *et al.*'s[19] value of r_c. Finally the stability condition will be discussed from the points of view of the vanishing $K_{\rm II}$ condition and on geometrical grounds based on the slope of the kink both leading to conditions on the non-singular remote stress term T.

THE MATHEMATICAL MODEL

In our description of the crack and the equations used, we will follow the convention in [14]. We consider the problem of a crack emanating from the free surface of a beam under remote pure bending as shown in Fig. 1 where the straight and the curved portions represent the initially long crack and its extension. The deviation of the crack from the x-

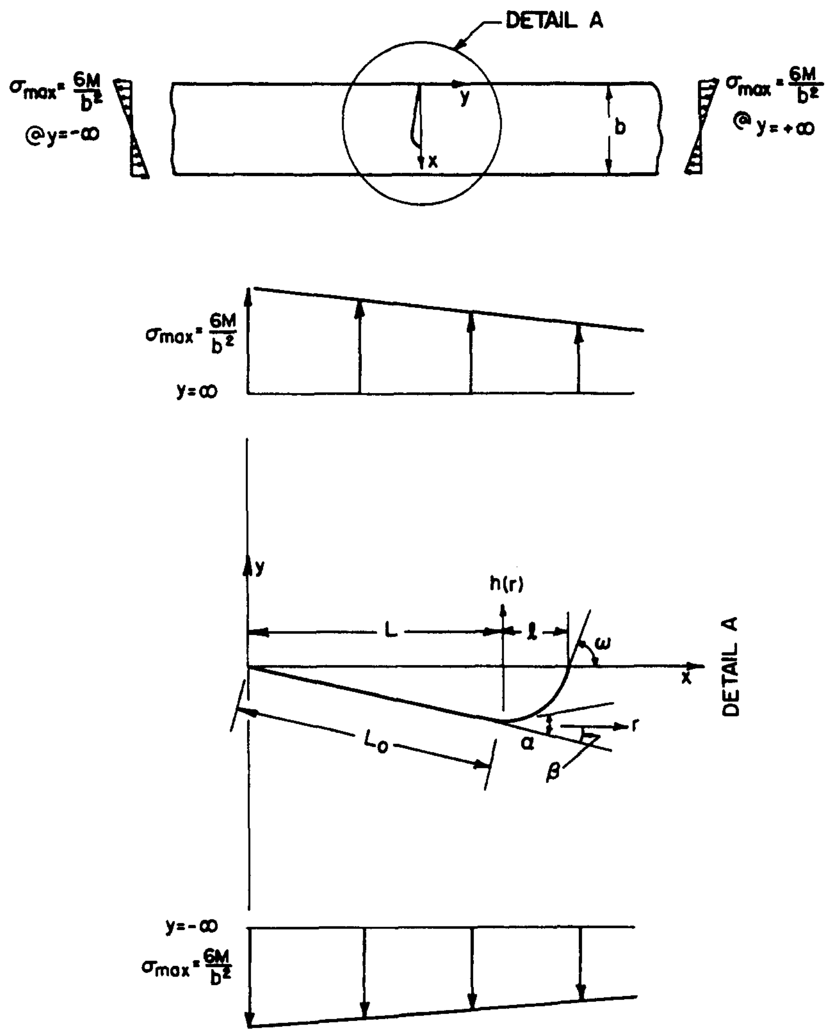

Fig. 1. Kinking crack in a beam under pure bending—with detail of kinking crack geometry.

axis is defined by $\Lambda(x)$ which is assumed to be small, relative to its extended length. The crack is assumed to be opened by surface normal and shear tractions T_n and T_s, which are necessary to remove the external tractions shown in Fig. 1.

We assume such a crack growth ($\beta \neq 0$) as an outcome of the possible nonalignment of the loading system (pure bending is achieved via four-point bending and one of the loads may not have been properly aligned) or the crack may not have been inserted initially perpendicular to the free surface.

As a prelude to the solution of our problem we solve a new problem depicted in Fig. 2, namely a finite crack in an infinite body antisymmetric with respect to the coordinate y. The problem is approached in this manner, since the solution of our problem may be derived from the solution of the new problem and since this will allow us to easily discern the effect of the free surface. Following [13], the stress field is obtained via stress functions $\phi(z)$ and $\psi(z)$ by means of Mushkelishvili's method[20]

$$\sigma_{xx}+\sigma_{yy} = 2[\phi(z)+\overline{\phi(z)}], \qquad \sigma_{yy}-\sigma_{xx}-2i\sigma_{xy} = 2[\overline{z\phi'(z)}+\overline{\psi(z)}] \tag{1}$$

where

$$z = x+iy, \quad i = \sqrt{-1}, \quad \bar{\phi}(\bar{z}) = \overline{\phi(z)} \quad \text{and} \quad \bar{\psi}(\bar{z}) = \overline{\psi(z)}.$$

The boundary condition on the crack is in the form

$$\phi(z)+\overline{\phi(z)}+\mathrm{e}^{-2i\theta}[\overline{z\phi'(z)}+\overline{\psi(z)}] = T_n-iT_s \tag{2}$$

where θ is the angle made by the crack with the x-axis ($\theta = \Lambda'(x) \ll 1$). In this problem, because of the symmetry in the stress field, viz. $\sigma_{mn}(z) = \sigma_{mn}(-z)$, this implies that $\phi(z) = \phi(-z)$ and $\psi(z) = \psi(-z)$.

Introducing the analytic function

$$\Omega(\bar{z}) = \overline{\phi(z)}+\overline{z\phi'(z)}+\overline{\psi(z)} \tag{3}$$

we obtain from (2)

$$\phi(z)+\overline{\phi(z)}+\mathrm{e}^{-2i\theta}[(z-\bar{z})\overline{\phi'(z)}+\Omega(\bar{z})-\overline{\phi(z)}] = T_n-iT_s. \tag{4}$$

A general overview of the perturbation scheme in [13] and found in more detail in [14] will be given. The reader is directed to these references for a more detailed description. We assume that there are two functions $F(z)$ and $W(z)$ corresponding to $\phi(z)$ and $\Omega(z)$ whose

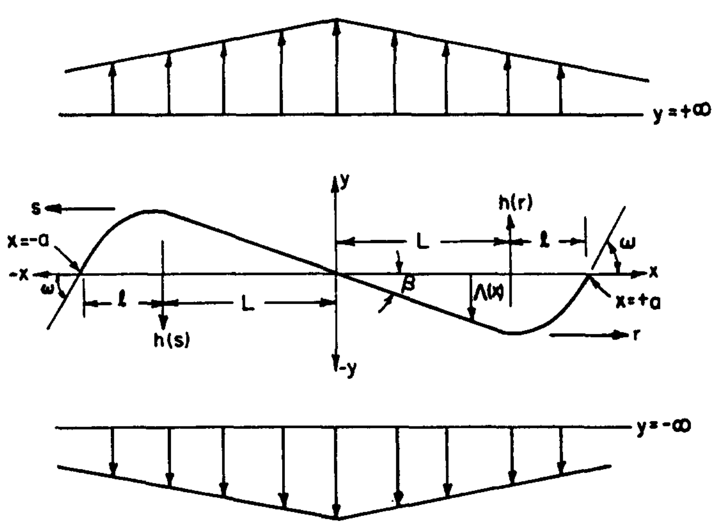

Fig. 2. Antisymmetric crack geometry in an infinite body.

boundary values are $F^{\pm}(t)$ and $W^{\pm}$ on the upper and lower surfaces of a straight cut located on the x-axis and in between the crack tips. The functions $\phi(z)$ and $\Omega(z)$ have their cut along the actual crack. The functions $F(z)$ and $W(z)$ can be written as follows retaining terms up to second order

$$\begin{aligned} F(z) &= \sum_{j=0}^{2} F_j(z) + 0(\Lambda^3) \\ W(z) &= \sum_{j=0}^{2} W_j(z) + 0(\Lambda^3) \end{aligned} \tag{5}$$

where the $F_j(z)$ and $W_j(z)$ are $0(\Lambda^j)$. It is assumed that these functions can be analytically continued to the faces of the actual crack. On the surfaces of the actual crack ($z = t + i\Lambda(t)$) $\phi^{\pm}(z)$ and $\Omega^{\pm}(z)$ can be expressed in terms of $F_j^{\pm}(t)$, $W_j^{\pm}(t)$ and their derivatives via a Taylor expansion in t using $i\Lambda(t)$ as the small parameter. For small Λ

$$e^{-2i\theta} \approx 1 - 2i\Lambda' - 2\Lambda'^2 + 0(\Lambda^3). \tag{6}$$

Employing (6) and the expansions for $\phi^{\pm}(z)$ and $\Omega^{\pm}(z)$ in (4) one obtains an equation for $T_n - iT_s$ in terms of Λ, $F_j^{\pm}(t)$, $W_j^{\pm}(t)$ and their derivatives. It was shown in [13] that the angle of the kinked extension was proportional to the ratio of the stress intensity factors K_{II}/K_I. As in [14] for the slightly deviated extension it is assumed that T_s is $0(\Lambda^1)$. This will then allow for the ordering of the terms of the expansions after their substitution in eqn (4) as follows:

$$\Lambda^0: \quad F_0^{\pm}(t) + W_0^{\mp}(t) = T_n \tag{7}$$

$$\Lambda^1: \quad F_1^{\pm}(t) + W_1^{\mp}(t) = -iT_s - i\Lambda[F_0^{\pm}(t) + W_0^{\mp}(t)]' - 2i\{\Lambda[\bar{F}_0^{\mp}(t) - W_0^{\mp}(t)]\}' \tag{8}$$

$$\begin{aligned} \Lambda^2: \quad F_2^{\pm}(t) + W_2^{\mp}(t) = &-i\Lambda[F_1^{\pm}(t) + W_1^{\mp}(t)]' - 2i\{\Lambda[\bar{F}_1^{\mp}(t) - W_1^{\mp}(t)]\}' \\ &+ \tfrac{1}{2}\Lambda^2[F_0^{\pm}(t) + W_0^{\mp}(t)]'' + 2[\Lambda^2\bar{F}_0^{\mp\prime}(t)]' - 2\Lambda'\{\Lambda[\bar{F}_0^{\mp}(t) - W_0^{\mp}(t)]\}' - 2\Lambda\Lambda'\bar{F}_0^{\mp\prime}(t). \end{aligned} \tag{9}$$

The solution of (7) as given by [20] via the boundary values of $[F_0(t) \pm W_0(t)]$ yields, with $a = L + l$,

$$F_0(z) = W_0(z) = \frac{1}{2\pi(z^2 - a^2)^{1/2}} \int_{-a}^{a} T_n \frac{(a^2 - t^2)^{1/2}}{t - z}\, dt. \tag{10}$$

The solution of (8) via its boundary values is

$$F_1(z) = W_1(z) = \frac{1}{2\pi(z^2 - a^2)^{1/2}} \int_{-a}^{a} -i(T_s + \Lambda T_n') \frac{(a^2 - t^2)^{1/2}}{t - z}\, dt; \tag{11}$$

and the solution of (9) via its boundary values is

$$F_2(z) + W_2(z) = \frac{1}{2\pi(z^2 - a^2)^{1/2}} \int_{-a}^{a} [2(\Lambda T_s)' + 2\Lambda' T_s + 4(\Lambda^2 T_n')' + \Lambda^2 T_n''] \frac{(a^2 - t^2)^{1/2}}{t - z}\, dt \tag{12}$$

as shown in [14].

Because of the symmetry condition on $\phi(z)$ and $\Omega(z)$ (hence on the $F_j(z)$ and $W_j(z)$'s) one obtains, if it is assumed that T_n and T_s are symmetric and Λ is antisymmetric in t,

$$
\begin{aligned}
F_0(-z) = W_0(-z) &= \frac{1}{2\pi(z^2-a^2)^{1/2}}\int_{-a}^{a} -T_n\frac{(a^2-t^2)^{1/2}}{t+z}\,\mathrm{d}t;\\
F_1(-z) = W_1(-z) &= \frac{1}{2\pi(z^2-a^2)^{1/2}}\int_{-a}^{a} i(T_s+\Lambda T_n')\frac{(a^2-t^2)^{1/2}}{t+z}\,\mathrm{d}t;\\
F_2(-z)+W_2(-z) &= \frac{1}{2\pi(z^2-a^2)^{1/2}}\int_{-a}^{a} -[2(\Lambda T_s)'+2\Lambda' T_s\\
&\quad +4(\Lambda^2T_n')'+\Lambda^2T_n''']\frac{(a^2-t^2)^{1/2}}{t+z}\,\mathrm{d}t.
\end{aligned}
\tag{13}
$$

These results can be obtained by inspection from (10)–(12) if one realizes that the same problem is solved when the axes are rotated 180° (defining the $-z$ coordinate system). The only difference is that the upper face of the crack in the z coordinate system is the lower face of the crack in the $-z$ coordinate system and vice versa. Now the tractions on the upper face in the z system are the negative of the tractions on the upper face of the $-z$ system, hence the sign change. The condition that T_n and T_s are symmetric in t is due to the loading conditions at infinity and the condition that the deviation of the crack from the x-axis is small.

If new functions $U_j(z)$ and $V_j(z)$ are defined as

$$
\begin{aligned}
U_j(z) &= \tfrac{1}{2}[F_j(z)+F_j(-z)]\\
V_j(z) &= \tfrac{1}{2}[F_j(z)+F_j(-z)]
\end{aligned}
$$

under the conditions that $T_n(t) = T_n(-t)$ and $T_s(t) = T_s(-t)$ then

$$
\begin{aligned}
U_0(z) = V_0(z) &= \frac{2z}{2\pi(z^2-a^2)^{1/2}}\int_{0}^{a} \frac{T_n(a^2-t^2)^{1/2}}{t^2-z^2}\,\mathrm{d}t;\\
U_1(z) = V_1(z) &= \frac{2z}{2\pi(z^2-a^2)^{1/2}}\int_{0}^{a} -i(T_s+\Lambda T_n')\frac{(a^2-t^2)^{1/2}}{t^2-z^2}\,\mathrm{d}t;\\
U_2(z)+V_2(z) &= \frac{2z}{2\pi(z^2-a^2)^{1/2}}\int_{0}^{a} [2(\Lambda T_s)'+2\Lambda' T_s\\
&\quad +4(\Lambda^2T_n')'+\Lambda^2T_n''']\frac{(a^2-t^2)^{1/2}}{t^2-z^2}\,\mathrm{d}t.
\end{aligned}
\tag{14}
$$

At a point $t = a+r$, the functions $F(z)$ and $W(z)$ are single-valued. Since $\Lambda(a) = 0$ then $\Lambda(a+r) = \Lambda'(a)r = \omega r$. Thus at such a point $\sigma_{\omega\omega}$ and $\sigma_{r\omega}$ may be obtained from (7)–(9) and (14).

Now for $z = t+i\Lambda(t)$ as z approaches a, one can define

$$
\begin{aligned}
K_{\mathrm{I}}-iK_{\mathrm{II}} &= \lim_{z\to a}\sqrt{2\pi(z-a)}\,(\sigma_{\omega\omega}-i\sigma_{r\omega}) = \lim_{r\to 0}\,(1+\omega^2)^{1/4}\cdot(2\pi r)^{1/2}\cdot(\sigma_{\omega\omega}-i\sigma_{r\omega})\\
&= (1+\omega^2)^{1/4}\cdot 2\sqrt{\frac{a}{\pi}}\int_0^a (q_{\mathrm{I}}-iq_{\mathrm{II}})\frac{\mathrm{d}t}{\sqrt{a^2-t^2}},
\end{aligned}
\tag{15}
$$

where

$$
q_{\mathrm{I}} = (5\omega^2/8-1)T_n+\tfrac{3}{2}\omega\Lambda T_n'-2(\Lambda^2T_n')'+\left(\frac{3\omega}{2}-\Lambda'\right)T_s-(\Lambda T_s)'-\tfrac{1}{2}\Lambda^2T_n''; \tag{16}
$$

$$q_{II} = -\left(T_s + \frac{\omega}{2}T_n + \Lambda T_n'\right). \tag{17}$$

Thus this is the solution for K_I and K_{II} for a finite crack in an infinite body that simultaneously kinks at both ends of the straight crack in an antisymmetric way as given in Fig. 2. To find the solution for a crack of the type shown in Fig. 1, we need only to remember that since T_n and T_s are symmetric in t and Λ is antisymmetric in t then $q_I(1+\omega^2)^{1/4}$ and $q_{II}(1+\omega^2)^{1/4}$ are symmetric functions of t. Also the integral in (15) may be viewed as an equivalent system of a straight crack between $x = a$ and $x = -a$ having symmetric loads $q_I(1+\omega^2)^{1/4}$ and $q_{II}(1+\omega^2)^{1/4}$. If so, Hartranft and Sih[17] have shown that knowledge of the solution for a straight crack in an infinite medium leads directly to the solution of a straight crack emanating from a free surface via

$$K_I - iK_{II} = 2\sqrt{\frac{a}{\pi}}\int_0^a (\sigma - i\tau)\cdot\frac{(1+f(t/a))}{\sqrt{a^2-t^2}}\,dt, \tag{18}$$

with

$$f(c = t/a) = (1-c^2)\,[0.2945 - 0.3912c^2 + 0.7685c^4 - 0.9942c^6 + 0.5094c^8]$$

and σ and τ being symmetric normal and shear tractions. Hence eqn (18) with σ and τ given by $(1+\omega^2)^{1/4}$ times eqns (16) and (17) yields the stress intensity factors for a straight crack emanating from a free surface that kinks as shown in Fig. 1.

From this point on the solution of the problem follows closely that found in [14]. Let L and l be the projections on the x-axis of the pre-existing crack (length L_0) and the extension (see Fig. 1). Letting $t = a - l + r$, we assume the crack path may be described by

$$\Lambda(t) = \lambda(r) = \begin{cases} h(r) - h(l) & 0 < r \leqslant l \\ -\beta(L+r) = -\beta(a-l+r) & -l \leqslant r \leqslant 0 \end{cases} \tag{19}$$

where

$$h(r) = (\alpha - \beta)r + \eta r^{3/2} + \chi r^2 \tag{20}$$

$$\beta = h(l)/L. \tag{21}$$

Even though $\Lambda(t)$ is antisymmetric in t, the patch function h in terms of the variable t is not. Hence any function in powers of $r^{1/2}$ may be used. Equation (20) is basically in the form of the solution obtained in [13] for a kink propagating from a semi-infinite crack in an infinite medium and is the same as eqn (24) in [14]. The coefficients α, η, χ are constants to be determined.

We integrate eqn (18) by parts and use Taylor's expansion on $(1+\omega^2)^{1/4}$. We next substitute for $a = L + l$, $(a-t) = l - r$ and $(a+t) = 2L + l + r$. We then expand $(L+l)^{1/2}$ and $(2L+l+r)^{1/2}$ in terms of r/L and l/L and assume that l/L is a small quantity of the same order as λ/l. By keeping terms up to second order the following is obtained

$$K_I = \sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_n\left\{1 - \frac{3}{8}\omega^2 + \frac{3}{2}\omega\lambda' + \frac{3\omega\lambda}{4(l-r)} + \lambda'^2 + \underline{\lambda\lambda''} + \frac{3\lambda\lambda'}{l-r} + \frac{15}{8}\frac{\lambda^2}{(l-r)^2}\right\}$$

$$\times\{1+f\}\frac{dr}{\sqrt{l-r}} - \sqrt{\frac{2}{\pi}}\int_{-L}^{l}\frac{T_n}{4L}(1+f)\sqrt{l-r}\,dr + \sqrt{\frac{2}{\pi}}\int_{-L}^{l}\frac{T_n}{32L^2}(1+f)$$

$$\times (5l+3r)\sqrt{l-r}\,\mathrm{d}r - \sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_s\left[\lambda' - \frac{3}{2}\omega - \frac{\lambda}{2(l-r)}\right][1+f]\frac{\mathrm{d}r}{\sqrt{l-r}}$$

$$- \sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_n\left\{\frac{3}{2}\omega\lambda + 6\lambda\lambda' + \frac{5}{2}\frac{\lambda^2}{l-r}\right\}\frac{f'\,\mathrm{d}r}{\sqrt{l-r}} - \sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_s\lambda f'\frac{\mathrm{d}r}{\sqrt{l-r}}. \tag{22a}$$

$$K_{II} = -\sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_n\left[\frac{\omega}{2} - \lambda' - \frac{\lambda}{2(l-r)}\right][1+f]\frac{\mathrm{d}r}{\sqrt{l-r}} - \sqrt{\frac{2}{\pi}}\int_{-L}^{l}\frac{T_n}{4L}$$

$$\times\left[\frac{\omega}{2} - \lambda' + \frac{\lambda}{2(l-r)}\right][1+f]\sqrt{l-r}\,\mathrm{d}r - \sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_s[1+f]$$

$$\times\left[1 + \frac{l-r}{4L}\right]\frac{\mathrm{d}r}{\sqrt{l-r}} + \sqrt{\frac{2}{\pi}}\int_{-L}^{l} T_n\lambda f'\left[1 + \frac{l-r}{4L}\right]\frac{\mathrm{d}r}{\sqrt{l-r}}. \tag{22b}$$

The primes in these equations denote differentiation with respect to r and f, f', f'' are given by

$$f = 0.374\cdot\left(\frac{l-r}{L}\right)\{1+0(l/L)\}$$

$$f' = \frac{1}{L}\cdot\{-0.374\cdot(1+0(l/L))\} \tag{23}$$

$$f'' = \frac{1}{L^2}\cdot\{-1.9804(1+0(l/L))\}.$$

The underlined term in (22a) is significant to the analysis and is discussed later in the text.

To simplify eqns (22), we resolve the component of the tractions T_n and T_s in the directions of the x- and y-axes and employ the small angle formulas, noting that $\Lambda'(t) = \lambda'(r)$, to obtain

$$\begin{aligned} T_n &= (1-\lambda'^2)\sigma_{yy} + \lambda'^2\sigma_{xx} + 2\lambda'\sigma_{xy} + 0(\lambda^3) \\ T_s &= (\sigma_{yy}-\sigma_{xx})\lambda' + (1-2\lambda'^2)\sigma_{xy} + 0(\lambda^3). \end{aligned} \tag{24}$$

In the case of crack extension the stresses on the boundary of the pre-existing crack are zero. This simplifies the integrals so that the lower limits are no longer $r = -L$ but $r = 0$. Note that by setting f, f', f'' equal to zero we obtain eqns similar to eqns (31) and (32) of [14]. They are not exactly the same due to the definition of L and the location of $\sqrt{a+t}$ in the integrand. The stresses on the extended portion may be obtained from the stress field that exists along the prolongation of the pre-existing crack tip [15]. In the r, h coordinate system the stresses on the curved extension may be expressed by:

$$\sigma_{mn}(r,h) = \sigma_{mn}(r,0) + h\frac{\partial}{\partial y}\sigma_{mn}(r,0) + \frac{h^2}{2}\frac{\partial^2}{\partial y^2}\sigma_{mn}(r,0) + 0(h^3) \quad m,n = x,y. \tag{25}$$

By employing the asymptotic expansions for the stress field together with its y-derivatives near the tip of the main crack, one finds that since the $B_j \sim 0(k_j/L)$ and K_{II} is $0(\lambda)$, then

$$\sigma_{xx}(r,h) = \frac{-k_I}{\sqrt{2\pi r}}\left(1 - \frac{9}{8}\frac{h^2}{r^2}\right) + \frac{3}{2}\frac{k_{II}}{\sqrt{2\pi r}}\left(\frac{h}{r}\right) - T - B_I\sqrt{\frac{r}{2\pi}} + 0(\lambda^3);$$

$$\sigma_{yy}(r,h) = \frac{-k_I}{\sqrt{2\pi r}}\left(1 + \frac{3}{8}\frac{h^2}{r^2}\right) - \frac{1}{2}\frac{k_{II}}{\sqrt{2\pi r}}\left(\frac{h}{r}\right) - B_I\sqrt{\frac{r}{2\pi}} + 0(\lambda^3);$$

$$\sigma_{xy}(r,h) = \frac{-k_I}{\sqrt{2\pi r}}\left(\frac{h}{2r} + \frac{\beta}{2}\right) - \frac{k_{II}}{\sqrt{2\pi r}} + T\beta + B_I\sqrt{\frac{r}{2\pi}}\frac{h}{2r} - B_{II}\sqrt{\frac{r}{2\pi}} + 0(\lambda^3). \tag{26}$$

The derivation of (26) is found in the Appendix of [14]. For the case of a moment on the crack a term linear in r also arises; however since we only keep terms to $0(\sqrt{r})$ it does not affect the computations. In these equations k_{I}, k_{II} are the Mode I and II stress intensity factors of the main crack (in Fig. 1), T is the non-singular constant term and B_{I}, B_{II} are the coefficients proportional to $\sqrt{r}$ in the Irwin–Williams expansions[10]. Again terms up to second order in λ are kept, consistent with the previous derivations. In these equations, β, given by (21), is assumed to be of second order in λ/l, i.e.

$$\frac{h(l)}{L} \cong \left[0\left\{\frac{\lambda(r)}{l} - \frac{h(r)}{l}\right\}\right] \cdot 0(l/L) = 0(\lambda^2/l^2).$$

By substituting (26) into (24), T_n and T_s are obtained. These are then substituted into (22) along with the definition of λ [from (19) and (20)] and $\omega(=\Lambda'(a) = h'(l))$. The resulting expressions define the stress intensity factors at the end of the kink (K_{I}, K_{II}) as functions of k_{I}, k_{II}, T, B_{I}, B_{II}, α, η, χ. Thus

$$\begin{aligned} K_{\mathrm{I}} = {} & k_{\mathrm{I}} - \frac{3}{8}\alpha^2 k_{\mathrm{I}} - \frac{3}{2}\alpha k_{\mathrm{II}} + \sqrt{l}\left\{k_{\mathrm{I}}\left(-\frac{9}{8}\eta\alpha + \frac{2}{\pi}\eta\alpha\right) - \frac{9}{4}\eta k_{\mathrm{II}} + 2\sqrt{\frac{2}{\pi}}\alpha^2 T \right. \\ & \left. - \frac{3}{2\pi}k_{\mathrm{I}}\eta\alpha \tanh^{-1}\sqrt{1-r^*/l} + \cdots\right\} + l\left\{k_{\mathrm{I}}\left(-\frac{9}{4}\chi\alpha - \left[\frac{39}{32} - \frac{6}{\pi}\right]\eta^2 + \frac{1}{8L}\right)\right. \\ & \left. - 3\chi k_{\mathrm{II}} + \frac{B_{\mathrm{I}}}{2} + \frac{11}{2}\sqrt{\frac{2}{\pi}}\eta\alpha T - \frac{3}{2\pi}k_{\mathrm{I}}\eta^2 \tanh^{-1}\sqrt{1-r^*/l} + \cdots\right\} \\ & + l^{3/2}\left\{k_{\mathrm{I}}\left(\frac{1}{2\pi} - \frac{9}{4}\right)\eta\chi + \sqrt{\frac{2}{\pi}}T\left(\left[\frac{39\pi}{16} - 4\right]\eta^2 + 7\alpha\chi\right) - \frac{3}{2\pi}k_{\mathrm{I}}\eta\chi\right. \\ & \left. \times \tanh^{-1}\sqrt{1-r^*/l} + \cdots\right\} + \left\{0.187 k_{\mathrm{I}}\frac{l}{L} + \cdots\right\} + 0(l^2); \end{aligned} \tag{27a}$$

$$\begin{aligned} K_{\mathrm{II}} = {} & k_{\mathrm{II}} + \frac{k_{\mathrm{I}}\alpha}{2} + \sqrt{l}\left\{\frac{3}{4}k_{\mathrm{I}}\eta - 2\sqrt{\frac{2}{\pi}}\alpha T + \cdots\right\} + l\left\{k_{\mathrm{I}}\left(\chi - \frac{\alpha}{16L}\right) + \frac{k_{\mathrm{II}}}{8L} + \frac{B_{\mathrm{II}}}{2}\right. \\ & \left. - \frac{B_{\mathrm{I}}\alpha}{4} - \frac{3\pi}{4}\sqrt{\frac{2}{\pi}}\eta T + \cdots\right\} + l^{3/2}\left\{\frac{-k_{\mathrm{I}}\eta}{32L} + \sqrt{\frac{2}{\pi}}T\left(\frac{8\chi}{3} - \frac{\alpha}{6L}\right) - \frac{\eta B_{\mathrm{I}}}{8} + \cdots\right\} \\ & + \left\{0.187\frac{l}{L}\left(\left[k_{\mathrm{II}} - \frac{k_{\mathrm{I}}\alpha}{2}\right] - 2\sqrt{l}\left[\frac{k_{\mathrm{I}}\eta}{8} + \frac{2}{3}\sqrt{\frac{2}{\pi}}\alpha T\right]\right) + \cdots\right\} + 0(l^2). \end{aligned} \tag{27b}$$

All terms kept are of zeroth, first or second order in l/L. The terms involving $\tanh^{-1}\sqrt{1-r^*/l}$ result from the underlined term in eqn (22a) and was not accounted for in eqn (35) of [14]. This term is not integrable at $r = 0$ but may be integrated up to a value of $r = r^*$. Since the stresses as defined in [10] or [18] involve a singularity, they will represent the actual stresses only in the vicinity of the crack and not at the tip. Thus the distance r^* can define the distance ahead of the main crack after which the stresses are actually represented by the Williams expansions. One may also look upon r^* as a distance where fracture initiates ahead of the crack[9] or a zone where voids and micro cracks grow and coalesce and, if the right conditions exist, can divert the crack from its original path[6, 7, 11, 12]. Thus r^* may be viewed in the same manner as Δa_c in [7], α_0 in [18] and r_c in [9, 11, 12]. The quantity r^*/l will be obtained from the ordering scheme.

Aside from the $\sqrt{1-r^*/l}$ terms and the terms due to the free surface (terms involving the coefficient 0.187), eqns (27) are similar to (35) and (36) in [14]. Some of the differences in the equations are due to L and $\sqrt{a+t}$ as discussed previously and some are due to algebraic errors in [14].

If α, η, χ in (27) are set equal to zero then

$$K_I = k_I + \frac{k_I}{8}\frac{l}{L} + \frac{B_I l}{2} + 0.187\frac{l}{L}k_I + 0(l^2)$$
$$K_{II} = k_{II} + \frac{k_{II}}{8}\frac{l}{L} + \frac{B_{II} l}{2} + 0.187\frac{l}{L}k_{II} + 0(l^2) \tag{28}$$

and as in [14] we can identify the crack extension change (as $l \to 0$) of K_I and K_{II} as

$$\frac{k_I}{8L} + \frac{B_I}{2} + 0.187\frac{k_I}{L} = \lim_{l\to 0}\frac{K_I - k_I}{l} \equiv \frac{\partial k_I}{\partial L_0}$$
$$\frac{k_{II}}{8L} + \frac{B_{II}}{2} + \frac{0.187 k_{II}}{L} = \lim_{l\to 0}\frac{K_{II} - k_{II}}{l} \equiv \frac{\partial k_{II}}{\partial L_0}. \tag{29}$$

The values for $\partial k_j/\partial L_0$ can be obtained via the boundary conditions thus enabling eqns (27) to be written as

$$\begin{aligned} K_I = {} & k_I - \frac{3}{8}\alpha^2 k_I - \frac{3}{2}\alpha k_{II} + \sqrt{l}\left\{k_I\left(\frac{2}{\pi} - \frac{9}{8}\right)\eta\alpha - \frac{9}{4}\eta k_{II} + 2\sqrt{2/\pi}\,\alpha^2 T \right. \\ & \left. - \frac{3}{2\pi}k_I\eta\alpha \tanh^{-1}\sqrt{1 - r^*/l} + \cdots\right\} + l\left\{\frac{\partial k_I}{\partial L_0} - k_I\left(\left(\frac{9}{4}\right)\chi\alpha + \left[\frac{39}{32} - \frac{6}{\pi}\right]\eta^2\right)\right. \\ & \left. - 3\chi k_{II} + \frac{11}{2}\sqrt{\frac{2}{\pi}}\eta\alpha T - \frac{3}{2\pi}k_I\eta^2 \tanh^{-1}\sqrt{1 - r^*/l} + \cdots\right\} + l^{3/2} \\ & \times\left\{k_I\left(\frac{1}{2\pi} - \frac{9}{4}\right)\eta\chi + \sqrt{\frac{2}{\pi}}T\left(\left[\frac{39\pi}{16} - 4\right]\eta^2 + 7\alpha\chi\right) - \frac{3}{2\pi}k_I\eta\chi \tanh^{-1}\sqrt{1 - r^*/l} + \cdots\right\} + 0(l^2). \end{aligned} \tag{30a}$$

$$\begin{aligned} K_{II} = {} & k_{II} + \frac{k_I\alpha}{2} + \sqrt{l}\left\{\frac{3}{4}k_I\eta - 2\sqrt{\frac{2}{\pi}}\alpha T + \cdots\right\} + l\left\{\frac{\partial k_{II}}{\partial L_0} - \frac{\alpha}{2}\frac{\partial k_I}{\partial L_0} + \chi k_I \right. \\ & \left. - \frac{3}{4}\pi\sqrt{\frac{2}{\pi}}\eta T + \cdots\right\} + l^{3/2}\left\{\sqrt{\frac{2}{\pi}}T\left(\frac{8\chi}{3} - \frac{\alpha}{6L}\right) - \frac{\eta}{4}\frac{\partial k_I}{\partial L_0} + \cdots\right\} - 0.187\frac{l^{3/2}}{L} \\ & \times\left\{\frac{4}{3}\sqrt{\frac{2}{\pi}}\alpha T + \cdots\right\} + 0(l^2). \end{aligned} \tag{30b}$$

As in [13, 14] we assume that crack extension will be in the direction of $K_{II} = 0$. Thus from (30b) the coefficients of the powers of l must be set to zero yielding

$$\alpha \cong -2k_{II}/k_I$$
$$\eta \cong \frac{8}{3}\left(\frac{2}{\pi}\right)^{1/2}\alpha T/k_I \tag{31a}$$
$$\chi \cong \alpha\left[4(T/k_I)^2 + \frac{1}{2k_I}\frac{\partial k_I}{\partial L_0} + \frac{1}{2k_{II}}\frac{\partial k_{II}}{\partial L_0}\right].$$

The same results found in [14]. However one also obtains from the $l^{3/2}$ coefficient that

$$\eta[4(T/k_I)^2 + \frac{1}{4k_I}\frac{\partial k_I}{\partial L_0} + \frac{1}{2k_{II}}\frac{\partial k_{II}}{\partial L_0} - \frac{1 + 4(0.374)}{16L_0} = 0. \tag{31b}$$

From $(31a)_2$ either α or T is zero or the term in the bracket is zero. The bracketed term set to zero is tantamount to defining a stability of crack path condition. In [6, 7] it is shown that even for some values of $T > 0$ the path is still stable.

Even though these results are obtained for the crack emanating from a free surface, multiplication of K_I and K_{II} by a dimensionless function of crack length will define the stress intensity factors for a crack in a beam under pure bending. However eqns (31) *will remain unaffected* (see Appendix).

Evaluation of r^*/l

We now turn our attention to the evaluation of r^*/l in (30a). To do this we must remember that l/L was assumed to be of the same order as λ/l. Because of this we can approximate our continuously turning crack by a straight crack of length $L_0+\varepsilon$ that grows a straight kink of length δ (see Fig. 3). It is assumed that the new approximation to our crack has the same intersection points on the x-axis and forms the same angle ω at P. If one performs the algebra the following is obtained

$$\begin{aligned} \frac{\overline{CA}}{L} &= \frac{\varepsilon}{L_0} \cong \frac{\varepsilon}{L} = \frac{l/L-\beta/\omega}{1+\beta/\omega} \cong \left(\frac{l}{L}\right)+0(\lambda^3) \\ \frac{\overline{PC}}{L} &= \frac{\delta}{L} \cong \frac{l}{L}\frac{(1-l/L)}{\cos\omega} \cong \frac{l}{L}\left(1-\frac{l}{L}\right)+0(\lambda^3), \end{aligned} \tag{32}$$

using the approximations for $\beta = \alpha l/L+0(\lambda^2)$ and $\omega = h'(l)$. For the straight crack OC that has deviated to point P we find that

$$\begin{aligned} K_I(P) &= \left(1-\frac{3}{8}\omega^2\right)K_I(C)-\frac{3}{2}\omega K_{II}(C)+\sqrt{\delta}\left[2\sqrt{\frac{2}{\pi}}\cdot\omega^2 T(C)\right]+0(\delta) \\ K_{II}(P) &= K_{II}(C)+\frac{\omega}{2}K_I(C)-\sqrt{\delta}\left[2\sqrt{\frac{2}{\pi}}\omega T(C)\right]+0(\delta). \end{aligned} \tag{33a}$$

These equations, which are similar to those in [6] with $k_{II} = 0$, are obtained via eqns (30) replacing α by $\omega+\beta$, setting η and χ to zero and realizing that the k_j's and T must be replaced by their corresponding values at point C. Now by using the Taylor expansion on $K_I(C)$, $K_{II}(C)$ and $T(C)$ in terms of their values at point A (via k_I, k_{II}, T) one will obtain

$$\begin{aligned} K_I(P) &= \left(1-\frac{3}{8}\omega^2\right)k_I-\frac{3}{2}\omega k_{II}+\sqrt{l}\left[2\sqrt{\frac{2}{\pi}}\omega^2 T\right]+0(l) \\ K_{II}(P) &= k_{II}+\frac{\omega k_I}{2}-\sqrt{l}\left[2\sqrt{\frac{2}{\pi}}\omega T\right]+0(l) \end{aligned} \tag{33b}$$

Equations (30) may also be written in terms of ω as

$$\begin{aligned} K_I &= \left(1-\frac{3}{8}\omega^2\right)k_I-\frac{3}{2}\omega k_{II}+\sqrt{l}\left[\underline{\frac{8}{3}\left(\frac{2}{\pi}-\frac{3}{2\pi}\tanh^{-1}\sqrt{1-r^*/l}\right)}+2\right]\cdot\sqrt{\frac{2}{\pi}}\omega^2 T+0(l); \\ K_{II} &= k_{II}+\frac{\omega k_I}{2}-\sqrt{l}\left(2\sqrt{\frac{2}{\pi}}\omega T\right)+0(l) \end{aligned} \tag{34}$$

where $(31a)_2$ was substituted for η and $\alpha^2 \cong \omega^2 + 0(\sqrt{l})$. Since (33b) and (34) are identical except for the underlined term, we set the term to zero yielding

$$\frac{r^*}{l} = 1 - \left[\tanh\left(\frac{4}{3}\right)\right]^2 \cong 0.243. \tag{35}$$

Thus for a given value of l/L, one then obtains r^*/L which can be compared to experimental values. It should be noted that since l/L is material, geometry and loading dependent so must r^*/L.

COMPARISON OF r^*/L WITH EXPERIMENTAL RESULTS

Even though a crack emanating from a free surface under a linearly varying far field stress was investigated, the method may also be applied to different far field loading conditions (e.g. a constant far field stress) provided one is careful about the terms kept in the expansions. One will notice that to $0(\lambda^2)$ the resulting equations for $K_{\rm I}$ and $K_{\rm II}$ are basically the same as those found by [14], the free surface conditions only playing a role in the redefinition of the k_j's, B_j's and $\partial k_j/\partial L_0$'s. As an outcome of this we can therefore compare our results to those of Ramulu *et al.*[19], whose dynamic analysis was performed on an SEN polycarbonate material under tension at essentially constant velocity.

Based on Fig. 3, we assume that we can model the turned crack by a straight kink whose turning angle is $\omega + \beta$. This angle is considered to be the branching angle θ_c in Table 1 of [19], point P being any point on the branched crack so that l/L is small. Because we are working with a branched crack $k_{\rm II}$ and $B_{\rm II}$ must be set to zero. By drawing a line between point P and the crack intersection with the free surface (not the tip of the starter crack), we can define the angle β. By using the definitions of β and ω as functions of (λ/l) we find that

$$\beta = \frac{-1 + \sqrt{1 + 4\theta_c}}{2}, \tag{36}$$

and, by the small angle approximation for β noting that ε/L is $0(l/L)^2$, we obtain

$$\tan\theta_c = \left(1 + \frac{L}{l}\right)\beta \cong \alpha\left(1 + \frac{l}{L}\right). \tag{37}$$

This leads to l/L, and, when multiplied by r^*/l from (35), yields r^*/L. In this way we may also obtain the predicted value of α. The results are summarized in Table 1. As one can see from the results, the predicted angles are in very good agreement with those found from experimental observation within the experimental scatter indicated, the worst result being the second entry where the starter crack was much smaller. We note that for almost constant velocity the ratio of r^*/r_c (process zone as found from (35) to the critical material parameter defined in [11, 12, 19]) is nearly a constant approximately equal to 3.7. The reason for the difference between r^* and r_c is due to the static analysis of our investigation

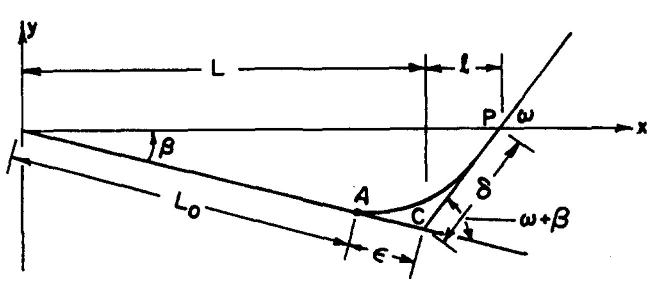

Fig. 3. Equivalent straight kink for the kinking curve.

Table 1. Comparison of experimental data[19] vs theoretical results

Test No.†	Branching angle (°) Measured†	Predicted via eqn (37)	c/c_1†	l/L	β	r_c/L†	r^*/L	r^*/r_c
KB-820826-1	34	33.3	0.22	0.161	5.35°	0.00988	0.03906	3.952
KB-820816-2	22	18.9	0.23	0.223	4.22°	0.015	0.0541	3.608
KB-820822	29	26.9	0.23	0.183	4.91°	0.012	0.0444	3.702
KB-820824	25	22.2	0.22	0.204	4.52°	0.014	0.0495	3.502
	Average 25		0.23					3.691 ± 7%

† Reference [19]—the measured angle has a scatter of at least ±12%.

and the dynamic investigation used for comparison. As shown in [12] the material parameter at zero crack tip velocity is higher than that at a non-zero velocity as noted by this ratio. The value of 3.7 is related to the velocity through the ratio of the static to the dynamic stress intensity factor, the remote non-singular stress component T as well as the crack tip velocity function[12]. It was quite fortuitous for us that these three parameters in these experiments were nearly constant in order for this comparison to be made.

PREDICTION OF CRACK PATH STABILITY CONDITION

We now turn our attention to eqn (31b) and the crack path stability condition. If we omit the 4(0.374) factor from the last term of the equation, the left-hand side of the equation will be the coefficient of the $l^{3/2}$ term of eqn (30b) for a finite kink emanating from a much longer straight crack in an infinite medium. This equation may be used for the problem studied in [14] provided $k_{\rm I}$, $k_{\rm II}$ and T are properly defined. For the case studied in [14] and using their notation

$$k_{\rm I} = \sigma_{\max}\sqrt{\pi L_0}, \quad k_{\rm II} = -\beta k_{\rm I} = -\frac{h(l)}{L_0}k_{\rm I}, \quad T = (k-1)\sigma_{\max}. \tag{38a}$$

Placing these into (31b) without the 4(0.374) factor yields

$$T^* = 0.384\sigma_{\max} \quad \text{or} \quad k^* = 1.384\sigma_{\max}. \tag{38b}$$

Thus for loadings in the direction of the crack greater than $1.384\sigma_{\max}$ the crack will turn. This is in physical agreement with the $k(=R)$ values of the crack paths shown in [13]. This value of T^* is also comparable to that found by Schindler and Sayir[7] via an energy approach. Even though they were investigating crack path stability in a beam in bending the crack geometry and near tip stress field employed there were basically the same as the one employed here. Also the crack tip velocity effect on the stress intensity factors was small at the point of path instability (crack speed/longitudinal wave speed, c/c_1, is about 0.1).

If the free surface effect is included in the pure bending case, placing in (31b)

$$k_{\rm I} = 1.122\sigma_{\max}\left(1-\frac{4\xi}{\pi}\right)\sqrt{\pi L_0}, \quad k_{\rm II} = -\beta k_{\rm I} \quad \text{and} \quad T = -1.122(1-2\xi)\sigma_{\max}, \tag{39a}$$

which are obtained from [21] and the Williams expansion for the problem, yields

$$\xi = \xi^* = \frac{L_0^*}{b} = 0.65 \quad \text{and} \quad T^* = 0.337\sigma_{\max}. \tag{39b}$$

This value of L_0^*/b represents the location of the instability of the path, L_0 and b representing the crack length and depth of the beam respectively. The justification of this formulation lies in the fact that for $L_0/b = 0.51$, $T\sqrt{L_0}/k_{\rm I} = 0.032$ which is in excellent agreement with the finite element solution of Larson and Carlsson[22] of an ASTM standard bend specimen with $L_0/b = 0.5$.

Thus it appears that the results of Cottrell and Rice[13] and Karihaloo *et al.*[14] on the stability of the crack path are restrictive for our case and may be modified as shown above. To verify this we need only look at the slope of the kink

$$h'(r) = \alpha - \beta + \tfrac{3}{2}\eta r^{1/2} + 2\chi r. \tag{40}$$

The angle α may be viewed as a parameter that characterizes the inability to set up the "perfect experiment", e.g. loads are located asymmetrically about the crack or where the initial crack cannot be "grown" perfectly perpendicular to the free surface, giving rise to a non-zero k_{II}. Thus we may speak of the deviatoric behavior of the kink's slope from straightness (given by $h'(r) = \alpha - \beta$ with respect to the x–y coordinate system) as unstable if the slope continuously increases. This is the same as requiring that

$$\tfrac{3}{2}\eta r^{1/2} + 2\chi r > 0 \qquad \forall r > 0. \tag{41}$$

If we only look at the η term, (41) would translate to the requirement that $T > 0$[13, 14]. However if we check the full inequality we find that for the beam in pure bending

$$\alpha \cdot \frac{4\sqrt{2}}{\pi\left(1 - \dfrac{4\xi}{\pi}\right)^2} \cdot \sqrt{\frac{r}{L_0}} \cdot \left[-(1-2\xi)\left(1 - \frac{4\xi}{\pi}\right) + \sqrt{\frac{2r}{L_0}} \left\{ (1-2\xi)^2 - \xi\left(1 - \frac{4\xi}{\pi}\right) \right\} \right] > 0. \tag{42a}$$

By setting the left-hand side of (42a) equal to zero there exists an r/L_0 value for which the path will become *stable* and this value is found by setting the bracketed term equal to zero. For the path to be unstable that implies that χ (in (41)) must be zero, requiring that as a minimum $\xi = \xi^* = L_0/b \cong 0.662$ and that $T^* = -1.122(1-2\xi)\sigma_{max} \cong 0.364\sigma_{max}$. This compares very well with Schindler and Sayir's values. Thus for values of $\frac{1}{2} \leqslant \xi < 0.662$ (or values of $0 \leqslant T < 0.364\sigma_{max}$) the crack path will still be stable for a crack past the centerline of the beam.

If eqn (41) were applied to the case investigated in [14] then

$$\alpha \left\{ 4(T/k_I) \sqrt{\frac{2r}{\pi}} [1 + \sqrt{2\pi r}(T/k_I)] \right\} > 0. \tag{42b}$$

Thus T would have to be greater than zero for unstable extension as suggested by [13, 14] and no value of r would cause the bracketted term to be zero or negative unless T itself were less than zero. The latter condition is precisely what Ramulu *et al.*[19] found for the case of an SEN tension plate experiment. Thus this physical requirement on the slope appears to corroborate the results of the stability condition given by (31b) and further extend it.

Another interesting aspect of eqn (41) is that it also predicts that for initially very long cracks in a beam under bending, crack extension will be stable. It can be seen that for $\xi > \pi/4$, the bracketed term in (42a) can be made less than or equal to zero for proper choice of r/L_0. This aspect is confirmed by experimentally-observed results for beams in pure bending with very low bending moments or initially long cracks (i.e. $L_0/b \approx 1$) [7, 23].

CONCLUSIONS

By use of a perturbation technique on the Mushkelishvili stress functions, the stress intensity factors for a crack emanating from a free surface and developing a kink were obtained for the case of remote pure bending of a beam. It was shown that as in [14], the crack path is dependent not only on the remote stress T, $T = -1.122[1-2(L_0/b)]\sigma_{max}$, and on the derivatives of k_I, k_{II} with respect to the pre-existing length L_0 but also implicitly upon the finite geometry correction. From the perturbation methodology a parameter r^*/l was obtained—a parameter which defined the limits of the Williams' stress field expansions and a necessary condition for crack curving. It was found to correlate well with available experimental data. Also provided by the methodology was the condition for unstable crack growth. It was shown that the Cottrell–Rice stability condition $T > 0$ was too restrictive for the beam bending case and that corroboration of this result was provided by looking at the deviation of the slope of the kink from straightness.

For the loading condition $k_{11} = 0$ and $\partial k_{11}/\partial L_0 \neq 0$ the path of the kink is found to be

$$h(r) = -\beta r + \frac{\beta r^2}{2L_0}\left[1 - \frac{2}{1 - \frac{4\xi}{\pi}}\right] + 0(r^{5/2})$$

so that without an initial kink the crack may have a smooth curving path in a stress field whose value of k_{11} changes with distance from the crack tip. One sees this in the experimental results of Kinra and Kolsky[1]. For values of ξ close to $\pi/4$ the change in slope is rather abrupt appearing as if the crack took a sharp right angle turn, again as observed in [1].

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REFERENCES

1. V. K. Kinra and H. Kolsky, The interaction between bending fractures and the emitted stress waves. *Engng Fract. Mech.* **9,** 423 (1977).
2. L. B. Freund and G. Herrmann, Dynamic fracture of a beam or plate in plane bending. *J. appl. Mech.* **43,** 112 (1976).
3. H. Adeli, G. Herrmann and L. B. Freund, Effect of axial force on dynamic fracture of a beam or plate in pure bending. *J. appl. Mech.* **44,** 647 (1977).
4. C. Levy and G. Herrmann, Effect of shear and rotary inertia on dynamic fracture of a beam or plate in pure bending. *J. appl. Mech.* **49,** 773 (1982).
5. C. Levy and G. Herrmann, On the effect of axial force on dynamic fracture of a beam or a plate in pure bending. *J. appl. Mech.* **49,** 656 (1982).
6. H. J. Schindler, Path stability of a crack crossing a beam in bending. *J. appl. Math. Phys.* (*ZAMP*) **32,** 570 (1981).
7. H. J. Schindler and M. Sayir, Path of a crack in a beam due to dynamic flexural fracture. *Int. J. Fract.* **25,** 95 (1984).
8. S. Bodner, Stress waves due to fracture of glass in bending. *J. Mech. Phys. Solids* **21,** 1 (1973).
9. R. Streit and I. Finnie, An experimental investigation of crack-path directional stability. *Exp. Mech.* **20,** 17 (1980).
10. M. L. Williams, On the stress distribution at the base of a stationary crack. *J. appl. Mech.* **24,** 109 (1957).
11. M. Ramulu, A. S. Kobayashi and B. S.-J. Kang, Dynamic crack curving and branching in line-pipe. Paper No. 82-WA/PVP-14, presented at the Winter Annual Meeting of the ASME, Phoenix, Arizona, 14–19 Nov. (1982).
12. M. Ramulu and A. S. Kobayashi, Dynamic crack curving—a photoelastic evaluation. *Exp. Mech.* **23,** 1 (1983).
13. B. Cottrell and J. R. Rice, Slightly curved or kinked cracks. *Int. J. Fract. Mech.* **16,** 155 (1980).
14. B. L. Karihaloo, L. M. Keer, S. Nemat-Nasser and A. Oranratnachai, Approximate description of crack kinking or curving. *J. appl. Mech.* **48,** 515 (1981).
15. N. V. Banichuk, Determination of the form of curvilinear crack by small parameter technique. *Izv. AN.SSR. M.T.T.* (*Mech. Solids*) **7,** 130 (1970) (in Russian).
16. R. V. Goldstein and R. L. Salganik, Brittle fracture of solids with arbitrary cracks. *Int. J. Fract. Mech.* **10,** 507 (1974).
17. R. J. Hartranft and G. C. Sih, Alternating method applied to edge and surface crack problems. *Mechanics of Fracture I—Methods of Analysis and Solutions of Crack Problems* (Edited by G. C. Sih), Chapter 4. Noordhoff International, The Netherlands (1973).
18. J. Eftis, N. Subramonian and H. Liebowitz, Crack border stress and displacement equations revisited. *Engng Fract. Mech.* **9,** 189 (1977).
19. M. Ramulu, A. S. Kobayashi, B. S.-J. Kang and D. B. Barker, Further studies on dynamic crack branching. Technical Report No. UWA/DME/TR-82/46 under ONR Contract No. N00014-76-0060 NR 064-478 (March 1983).
20. N. I. Mushkelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 3rd edn. Noordhoff, The Netherlands (1953).

21. H. Tada, P. Paris and G. Irwin, *The Stress Analysis of Cracks Handbook*. Del Research Corp (1973).
22. S. G. Larson and A. J. Carlsson, Influence of non-singular stress terms and specimen geometry on small scale yielding of the crack tip in elastic–plastic materials. *J. Mech. Phys. Solids* **21**, 263 (1973).
23. G. Herrmann, Private communication (1978).

APPENDIX

Since it was assumed that the cut between $x = 0$ and $x = a$ (Fig. 1) represented the effective crack with tractions $q_{I}(1+\omega^2)^{1/4}$ and $q_{II}(1+\omega^2)^{1/4}$, we can multiply the resulting stress intensity factors, eqn (27), by a dimensionless function $g(\xi)$, ξ being the crack length to beam height ratio, which represents the finite geometry correction. Since each term on the right-hand side of (27) is multiplied by $g(\xi)$, only the k_j's, B_j's (hence $\partial k_j/\partial L_0$'s) and T will be affected. However *ratios* of these quantities will *not* be affected. Thus $(31a)_1$ and $(31a)_2$ will be unaffected.

Equation $(31a)_3$ will then become

$$\chi_{new} = \alpha\left[\frac{1}{2\bar{k}_I}\frac{\partial \bar{k}_{II}}{\partial L_0} + \frac{1}{2\bar{k}_{II}}\frac{\partial \bar{k}_{II}}{\partial L_0} + 4\left(\frac{\bar{T}}{\bar{k}_I}\right)^2 - \frac{\xi}{L_0}\cdot\frac{g'(\xi)}{g(\xi)}\right] \tag{A1}$$

where

$$\bar{k}_j = k_j\cdot g(\xi) \quad \text{and} \quad \bar{T} = T\cdot g(\xi). \tag{A2}$$

Since

$$\frac{1}{\bar{k}_j}\frac{\partial \bar{k}_j}{\partial L_0} = \frac{1}{k_j}\frac{\partial k_j}{\partial L_0} + \frac{\xi}{L_0}g'(\xi)/g(\xi) \tag{A3}$$

then

$$\chi_{new} = \chi = [\text{eqn } (31a)_3].$$

Similarly, the stability criterion will take the form

$$\eta\left[4(\bar{T}/\bar{k}_I)^2 + \frac{1}{2\bar{k}_{II}}\frac{\partial \bar{k}_{II}}{\partial L_0} + \frac{1}{4\bar{k}_I}\frac{\partial \bar{k}_I}{\partial L_0} - \frac{1+4(0.374)}{16L_0} - \frac{3}{4}\frac{\xi}{L_0}\frac{g'(\xi)}{g(\xi)}\right] = 0. \tag{A4}$$

When (A2) is put in (A4) the resulting equation is identical to (31b). The outcome of these results is that the kink path, to second order, is unaffected by the finite geometry corrections $1.22\cdot g(\xi)$. Also r^*/l, to first order, will not be affected either. However the non-singular term T will be and it is necessary to evaluate $g(\xi)$.

To obtain $g(\xi)$ let us identify the material toughness, K_{I_c}, to be $\bar{k}_I$ and define a hypothetical dimensionless "starter" crack length ξ_f [2–5] so that

$$K_{I_c} \equiv 1.122\sigma_{max}\sqrt{\pi\xi_f b} = \bar{k}_I = 1.122\sigma_{max}\sqrt{\pi\xi b}\left(1 - \frac{4\xi}{\pi}\right)g(\xi). \tag{A5}$$

Then

$$g(\xi) = \frac{\sqrt{\xi_f}}{\sqrt{\xi}\left(1 - \frac{4\xi}{\pi}\right)} \tag{A6}$$

and

$$T = -1.122(1-2\xi)g(\xi)\sigma_{max}. \tag{A7}$$

At the kink location, the global definition of T given by (A7) must be equal to the local definition of T, given by (39a) in the text, requiring that

$$g(\xi = \xi^*) = 1 \tag{A8}$$

and

$$T^* = 0.364\sigma_{max} \quad \text{at} \quad \xi = \xi^* = 0.662. \tag{A9}$$

The dimensionless length of the "starter" crack from (A6) must be

$$\xi_f \cong 0.016. \tag{A10}$$

This implies that if the bending moment causing fracture is reduced by one-half, the starter crack quadruples and, for the same critical value of T^*, the crack will kink farther into the beam at about $\xi = 0.75$. This is in very good agreement with the results found in [7]. We may therefore conclude that the function $g(\xi)$ only plays a part in relating beam bending experiments with different values of σ_{max} but does not actually play a role in defining T^*.